Tutorial 8

March 23, 2017

1. **Method of characteristic coordinates on p71**
   Theorem 1(on P69): The unique solution of

$$\begin{cases} \partial_t^2 u - c^2 \partial_x^2 u = f(x,t), -\infty < x < \infty, t > 0 \\ u(x, t=0) = \phi(x), -\infty < x < \infty \\ \partial_t u(x, t=0) = \psi(x), -\infty < x < \infty \end{cases}$$

is

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy + \frac{1}{2c}\iint_\Delta f(y,s)dyds$$

where Δ is the characteristic triangle.

Proof: Use the characteristic coordinates:

$$\xi = x + ct$$

$$\eta = x - ct$$

Then we have

$$Lu = \partial_t^2 u - c^2 \partial_x^2 u = -4c^2 \partial_{\xi\eta}^2 u = f(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c})$$

We integrate this equation with respect to η, leaving ξ as a constant.Thus $\partial_\xi u = -\frac{1}{4c^2}\int^\eta f d\eta$. Then we integrate with respect to ξ to get

$$u = -\frac{1}{4c^2}\int^\xi \int^\eta f d\eta d\xi$$

The lower limits of integration here are arbitrary: They correspond to constants of integration.Now we make a particular choice of the lower limits and find a particular solution:

$$\begin{aligned} u(P_0) = u(x_0, y_0) &= -\frac{1}{4c^2}\int_{\eta_0}^{\xi_0}\int_{\xi}^{\eta_0} f(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c})d\eta d\xi \\ &= \frac{1}{4c^2}\int_{\eta_0}^{\xi_0}\int_{\eta_0}^{\xi} f(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c})d\eta d\xi \\ &= \frac{1}{4c^2}\int_0^{t_0}\int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} f(x,t)Jdxdt \quad \text{(change of variables)} \\ &= \frac{1}{4c^2}\int_0^{t_0}\int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} f(x,t)2cdxdt \\ &= \frac{1}{2c}\iint_\Delta f(x,t)dxdt \end{aligned}$$

This is precisely Thereom 1. (See the figures on the books and you will understand clearly.)

2. Show that Robin boundary conditions are symmetric.

Solution: Suppose f and g are two functions satisfying Robin conditions

$$X'(0) - a_0 X(0) = 0, X'(l) + a_l X(l) = 0$$

then

$$\begin{aligned} f'\bar{g} - f\bar{g}'\Big|_0^l &= f'(l)\overline{g(l)} - f(l)\overline{g'(l)} - f'(0)\overline{g(0)} + f(0)\overline{g'(0)} \\ &= -a_l f(l)\overline{g(l)} + a_l f(l)\overline{g(l)} - a_0 f(0)\overline{g(0)} + a_0 f(0)\overline{g(0)} = 0 \end{aligned}$$

3. (Exercise 4 on P129: MSC $\not\Rightarrow$ PC)

Let

$$g_n(x) = \begin{cases} 1 \text{ in the interval } \left[\frac{1}{4} - \frac{1}{n^2}, \frac{1}{4} + \frac{1}{n^2}\right) & \text{for odd } n \\ 1 \text{ in the interval } \left[\frac{3}{4} - \frac{1}{n^2}, \frac{3}{4} + \frac{1}{n^2}\right) & \text{for even } n \\ 0 & \text{for all other } x. \end{cases}$$

Show that $g_n(x) \to 0$ in the L^2 sense but that $g_n(x)$ does not tend to zero in the pointwise sense.

Solution: On the one hand,

$$\int_{-\infty}^{\infty} |g_n(x) - 0|^2 dx = \begin{cases} \displaystyle\int_{\frac{1}{4}-\frac{1}{n^2}}^{\frac{1}{4}+\frac{1}{n^2}} 1^2 dx = \frac{2}{n^2} \to 0 \quad n : \text{odd} \\ \displaystyle\int_{\frac{3}{4}-\frac{1}{n^2}}^{\frac{3}{4}+\frac{1}{n^2}} 1^2 dx = \frac{2}{n^2} \to 0 \quad n : \text{even} \end{cases} \quad \text{as} \quad n \to \infty$$

Hence $g_n(x) \to 0$ in the L^2 sense.

On the other hand, for all odd n, we have $g_n(\frac{1}{4}) = 1$ which implies that $g_n(x)$ cannot tend to zero in the pointwise sense.

4. Fourier series, Convergence theorems and their application.

First, the Fourier sine series of $\phi(x) = 1$ on $[0, \pi]$ is

$$\sum_{n=1}^{\infty} A_n \sin(nx)$$

where

$$A_n = \frac{2}{\pi} \int_0^{\pi} \phi(x) \sin(nx) dx = -\frac{2}{n\pi} \cos(nx)\Big|_0^{\pi} = \begin{cases} 0, & n: \text{ even} \\ \frac{4}{n\pi} & n: \text{ odd} \end{cases}$$

That is, the Fourier sine series is

$$\sum_{n=1;odd}^{\infty} \frac{4}{n\pi} \sin(nx) = \frac{4}{\pi}(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots).$$

Second, note that $\phi(x) = 1$ is continuous up to second derivatives on $[0, \pi]$, but ϕ does not satisfy the Dirichlet boundary conditions (for Fourier sine series), we can not derive the uniform convergence by uniform convergence theorem (on P124). However, the pointwise convergence theorem (on P125) is available, thus the Fourier sine series converge to ϕ pointwisely on the open interval $(0, \pi)$, which means

$$\sum_{n=1;odd}^{\infty} \frac{4}{n\pi} \sin(nx) = 1$$

for any $0 < x < \pi$.

Finally, for $x = \frac{\pi}{2} \in (0, \pi)$, we have

$$1 = \sum_{n=1;odd}^{\infty} \frac{4}{n\pi} \sin(nx) = \frac{4}{\pi}(1 - \frac{1}{3} + \frac{1}{5} - \cdots)$$

that is,

$$\frac{\pi}{4} = \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1}.$$

Similarly, for $x = 1 \in (0, \pi)$, we have

$$\frac{\pi}{4} = \sin 1 + \frac{1}{3} \sin 3 + \frac{1}{5} \sin 5 + \cdots .$$